ISSN: 2395-6607, Vol. 2, No. 1 March 2016, pp. 6–70

Remarks on Negacyclic Matrices

Jennifer Seberry*

January 16, 2016

Abstract

A matrix is *negacyclic* if it commutes with the matrix $P = (p_{ij})$, $p_{i,i+1} = 1, i = 0,\ 1,\ \cdots,\ n-2$; $p_{n-1,0} = -1$; $p_{ij} = 0$ otherwise. This remark discusses new *weighing matrices* and other orthogonal matrices constructed using negacyclic matrices. Examples are given and a number of unanswered research questions posed.

Keywords: *Negacyclic matrix, Hadamard matrix, orthogonal design, weighing matrices, circulant matrices, 05B20.*

1 Introduction

A type of weighing matrix, of weight n and weight $n-1$, called a C-matrix or conference matrix, was previously studied by Delsarte-Goethals-Seidel [4]. These can be based on circulant or on negacyclic matrices. We consider these negacyclic based matrices with weight $k \le n$.

Definition 1.1. Let P, called the "negacyclic shift matrix" be the square matrix of order n, whose elements p_{ij} are defined as follows:

$$\begin{aligned} p_{i,i+1} &= 1, \quad i = 0, 1, \ldots, n-2, \\ p_{n-1,0} &= -1, \\ p_{ij} &= 0, \quad \text{otherwise.} \end{aligned}$$

Any matrix of the form $\sum a_i P^i$, with a_i commuting coefficients, will be called *negacyclic*.

We see there are similarities but not necessarily sameness between the properties of circulant/cyclic matrices and negacyclic matrices.

Lemma 1.2. *Let $P = (p_{ij})$ of order n be a negacyclic matrix. Then*

*Centre for Computer and Information Security Research, EIS, University of Wollongong, NSW 2522, Australia. Email: jennifer_seberry@uow.edu.au

(i) *The inner product of the first row of* P *with the* i^{th} *row of* P *equals the negative of the inner product of the first row of* P *with the* $(-i+2)^{nd}$ *row. That is*

$$\sum_{j=1}^{n} p_{1j}p_{ij} = -\sum_{j=1}^{n} p_{1j}p_{n-i+2,j} \tag{1}$$

(This is the negative of the result for circulant/cyclic matrices).

(ii) *The inner product of the first row of* P *with the* i^{th} *row of* P *equals the inner product of the* k^{th} *row of* P *with the* $(i+k-1)^{st}$ *row of* P*. That is*

$$\sum_{j=1}^{n} p_{1j}p_{ij} = \sum_{j=1}^{n} p_{kj}p_{i+k-1,j} \tag{2}$$

(This is the same result as for circulant/cyclic matrices).

(iii) *Then* P *of order* n *satisfies*

$$P^n = -1, \quad P^\top = -P^{n-1}, \quad PP^\top = I\,.$$

If $A = \sum a_i P^i$, $B = \sum b_j P^j$ *and* R *is the back diagonal matrix, then*

$$AB = BA \ \text{ and } \ A(BR)^\top = BRA^\top\,.$$

A and BR are amicable matrices.

We now note some other properties of negacyclic matrices which were shown by L.G. Kovacs and Peter Eades [5]. The second result appears in Geramita and Seberry [8, 206-207]. We give the proof here to emphasize a result which appears to have been forgotten.

Lemma 1.3. *If* $A = \sum a_i P^i$ *is a negacyclic matrix of odd order* n*, then* XAX*, where* $X = diag(1,-1,1,-1,\ldots,1)$*, is a circulant matrix.*

Lemma 1.4. *Suppose* $v \equiv 0 \pmod 2$*. The existence of a negacyclic* $N = W(v, v-1)$ *is equivalent to the existence of a* $W(v, v-1)$ *of the form*

$$\begin{bmatrix} A & B \\ B^\top & -A^\top \end{bmatrix} \tag{3}$$

where A *and* B *are negacyclic of order* $\frac{1}{2}n$*,* $A^\top = (-1)A^{\frac{1}{2}n}A$*. That is the 2-block suitable matrix gives a weighing matrix which is equivalent to a 1-block matrix.*

Proof. First we suppose there is a negacyclic matrix $N = W(2n, 2n-1)$ of order 2 which is used to form two negacyclic matrices A and B of order $2n$ which satisfy

$$AA^\top + BB^\top = (2n-1)I. \tag{4}$$

Let the first row of the negacyclic matrix N be

$$0x_1y_1x_2y_2\ldots y_{n-1}x_n$$

We choose A and B to be negacyclic matrices with first rows

$$0y_1y_2\ldots y_{n-1}, \text{ and } x_1x_2\ldots x_n\,,$$

respectively. If the order $n = 2t+1$ is odd and the first rows of A and B are

$$0a_1\ldots a_t\,(\epsilon_t a_t)\ldots(\epsilon_1 a_1) \text{ and } 1b_1b_2\ldots b_t\,(\delta_t b_t)\ldots(\delta_1 b_1)\;,$$

with $\epsilon_i = \pm 1$, $\delta_j = \pm 1$, then taking the dot product of the first and $(i+1)^{\text{th}}$ rows, $i \le t$ (reducing using $xy \equiv x + y - 1 \pmod 4$), we obtain

$$2t - 2i + \epsilon_i \pmod 4 \text{ and } 2t - 2i + 1 \pmod 4\,,$$

respectively. Hence using equation (4),

$$\epsilon_i + 1 \equiv 0 \pmod 4\,,$$

we have $\epsilon_i = -1$.

If the order n is even and the first rows of A and B are

$$0a_1\ldots a_{t-1}a_t(\epsilon_{t-1}a_{t-1})\ldots(\epsilon_1 a_1)$$

and

$$1b_1b_2\ldots b_t(\delta_{t-1}b_{t-1})\ldots(\delta_1 b_1),$$

with $\epsilon_i = \pm 1$, $\delta_j = \pm 1$, then taking the dot product of the first and (i+1)$^{\text{th}}$ rows, $i \le t-1$ (reducing modulo 4), we obtain

$$2t - 2t - 1 + \epsilon_i \pmod 4 \text{ and } 2t - 2i + 2b_i - 2 \pmod 4\,,$$

respectively. Hence, using equation (4),

$$\epsilon_i + 2b_i - 3 \equiv 0 \pmod 4\,,$$

and since $b_i \ne 0$, we have $\epsilon_i = 1$.

This means the first row of the original negacyclic matrix of order $2n$ can be written as

$$0x_1a_1x_2a_2\ldots x_ta_t1\bar{a}_t(\delta_t x_t)\bar{a}_{t-1}\ldots\bar{a}_2(\delta_2 x_2)\bar{a}_1(\delta_1 x_1) \text{ for } n \text{ odd}$$

and

$$0x_1a_1x_2a_2\ldots a_{t-1}x_ta_t(\delta_t x_t)a_{t-1}\ldots a_2(\delta_2 x_2)a_1(\delta_1 x_1) \text{ for } n \text{ even}$$

with $\delta_j = \pm 1$ and $\bar{a}_i = -a_i$.

The inner product of the first and $(2i-1)^{\text{th}}$ rows, $i \le t$ and $t-1$ respectively, is

$$-\delta_i + 1 \equiv 0 \pmod 4 \text{ and } \delta_i + 1 \equiv 0 \pmod 4 .$$

So we have the first rows of A and B

$$0a_1 \dots a_t \bar{a}_t \dots \bar{a}_1 \text{ and } b_1 b_2, \dots b_t 1 b_t \dots b_2 b_1 \text{ for } n \text{ odd} \tag{5}$$

and

$$0a_1 \dots a_{t-1} a_t a_{t-1} \dots a_1 \text{ and } b_1 b_2 \dots b_t \bar{b}_t \dots \bar{b}_2 \bar{b}_1 \text{ for } n \text{ even} \tag{6}$$

as required.

It is straightforward to check that negacyclic matrices A and B, which satisfy $AA^\top + BB^\top = (2n-1)I_n$ and are of the form (5) and (6), give a negacyclic matrix $W(2n, 2n-1)$ when formed into first rows

$$0b_1 a_1 b_2 \dots b_t a_t 1 \bar{a}_t b_t \dots \bar{a}_1 b_1, \text{ for } n \text{ odd,}$$

or

$$0b_1 a_1 b_2 \dots b_t a_t \bar{b}_t \dots a_1 \bar{b}_1 \text{ for } n \text{ even.}$$ □

Example 1.5. The first rows of negacyclic matrices $(n, n-1)$ of orders 4, 6, 8, and 10, respectively;

$$\begin{array}{l} 0\,1\,1\,-, \\ 0\,1\,-\,1\,1\,1 \\ 0\,1\,1\,-\,1\,1\,1\,- \\ 0\,1\,1\,-\,1\,-\,-\,-\,-\,1. \end{array}$$

are equivalent to the existence of

$$\begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}, \begin{bmatrix} 1 & - \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & - & 1 \\ - & 0 & - \\ 1 & - & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ - & 1 & 1 \\ - & - & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ - & 0 & 1 & 1 \\ - & - & 0 & 1 \\ - & - & - & 0 \end{bmatrix}, \begin{bmatrix} 1 & - & 1 & - \\ 1 & 1 & - & 1 \\ - & 1 & 1 & - \\ 1 & - & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 & - & - \\ 1 & 0 & 1 & 1 & - \\ 1 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 0 & 1 \\ - & - & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & - & - & - & 1 \\ - & 1 & - & - & - \\ 1 & - & 1 & - & - \\ 1 & 1 & - & 1 & - \\ 1 & 1 & 1 & - & 1 \end{bmatrix}$$

Comment. Peter Eades [5] and Delsarte-Goethals-Seidel [4] have determined that the only negacyclic $W(v, v-1)$ of order $v < 1000$ have $v = p^r + 1$ where p^r is an odd prime power. On the positive side we know (we omit the proof):

Theorem 1.6 (Delsarte-Goethals-Seidel [4]). *There is a negacyclic* $W(p^r + 1, p^r)$ *whenever* p^r *is an odd prime power.*

G. Berman [2] has led us to believe that many results of a similar type to those found for circulant matrices can be obtained using negacyclic matrices. Negacyclic matrices are curiosities because of their properties and potential exhibited in Lemma 1.4 and Example 1.7.

Example 1.7. The four negacyclic matrices

$$A_1 = \begin{bmatrix} 1 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 0 \\ 0 & 0 & 1 & - & 0 \\ 0 & 0 & 0 & 1 & - \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 & - & - & 0 \\ 0 & 1 & 1 & - & - \\ 1 & 0 & 1 & 1 & - \\ 1 & 1 & 0 & 1 & 1 \\ - & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & - & 0 & 0 & 1 \\ - & 0 & - & 0 & 0 \\ 0 & - & 0 & - & 0 \\ 0 & 0 & - & 0 & - \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfy

$$A_1A_1^\top + A_2A_2^\top + A_3A_3^\top + A_4A_4^\top = 9I\,.$$

They can be merged to form two negacyclic matrices

$$B_1 = \begin{bmatrix} 1 & 0 & - & - & 0 & 0 & 0 & 0 & 0 & 1 \\ - & 1 & 0 & - & - & 0 & 0 & 0 & 0 & 0 \\ & & & & \text{etc.} & & & & & \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 0 & 1 & - & - & 0 & - & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & - & - & 0 & - & 0 & 0 \\ & & & & \text{etc.} & & & & & \end{bmatrix}$$

which satisfy

$$B_1B_1^\top + B_2B_2^\top = 9I\,.$$

These can be further merged to obtain the first row of a negacyclic $W(20, 9)$:

$$1\ 1\ 0\ 0\ -\ 1\ -\ -\ 0\ -\ 0\ 0\ 0\ -\ 0\ 0\ 0\ 0\ 1\ 0.$$

Negacyclic matrices are worthy of further existence searches. The question of when negacyclic matrices can be decomposed as in Example 1.7 is open for further research.

2 Constructions

Suitable (plug-in) matrices $X_1, X_2, X_3, X_4, \cdots X_t$ are t matrices of order n, with elements ± 1 which satisfy the additive property, $\sum_{i=1}^{t} X_i X_i^\top$ = constant times the identity matrix. They are suitable if they satisfy other equations which enable them to be substituted into a plug-into array to make an orthogonal matrix. Xia, Xia and Seberry [26] show 4-suitable plug-in negacyclic matrices of odd order exist if and only if 4-suitable plug-in circulant matrices exist for the same odd order. 4-suitable negacyclic matrices of order n, may be used instead of 4-suitable circulant matrices, in the Goethals-Seidel plug-into array [9], construct Hadamard matrices of order $4n$. Other useful plug-into arrays are due to Kharaghani, Ito, Spence, Seberry-Balonin and Wallis-Whiteman [18, 15, 16, 22, 21, 24].

In computer searches, for some even orders, 2-suitable or 4-suitable negacyclic matrices have proved easier to find. This experimental fact has been used extensively by Holzmann, Kharaghani and Tayfeh-Rezaie [13, 14, 7, 6, 20] to complete searches for OD's in orders 24, 46, 48, 56, and 80. We note that if there are 2-suitable negacyclic matrices of order n and Golay sequences of order m, there are 2-suitable matrices of order mn.

This means a negacyclic matrix may give 2-suitable and 4-suitable plug-in matrices to use in plug-into arrays to make larger orthogonal matrices.

From Table 1 there exist $W(12, k)$ constructed using two negacyclic matrices of order 6 for $k = 1, 2, 4, 5, 6, 7, 8, 10, 12$. From Delsarte-Goethals-Seidel [4], there exists $0, \pm 1$ negacyclic $W(12, 11)$. From Table 2 there exists $W(12, k)$ constructed via 4 negacyclic matrices of order 3 for $k = 1, 2, \ldots, 12$.

Table 1: First rows of $W(12, k)$ constructed from two negacyclic matrices of order 6

k	First Rows
1	$1\,0\,0\,0\,0\,0\,;\,0_6$
2	$1\,0_5\,;\,1\,0_5$
3	$1\,0\,0\,1\,0\,;\,1\,0_5$
4	$1\,1\,0_4\,;\,1\,-\,0_4$
5	$1\,1\,-\,0_3\,;\,1\,0\,1\,0_3$
6	$0\,1\,1\,1\,-\,1\,;\,0_6$
6	$0\,1\,1\,1\,-\,1\,;\,1\,0_5$

k	First Rows
7	$0\,1\,1\,1\,-\,1\,;\,1\,0\,0\,1\,0\,0$
8	$1\,1\,-\,1\,0_4\,;\,1\,1\,1\,-\,0_4$
9	? ; ?
10	$0\,1\,1\,1\,-\,1\,;\,0\,1\,1\,1\,-\,1$
12	$1\,1\,1\,1\,-\,1\,;\,-\,1\,1\,1\,-\,1$

Remark. The question of which $W(4n, k)$ can be constructed using two negacyclic $0, \pm 1$ matrices of order $2n$ has yet to be resolved.

It is easy to see that there exist $W(2n, k)$ constructed from 2 negacyclic matrices of order n whenever there exist two 0, ± 1 sequences of length n and weight k with NPAF zero.

Table 2: First rows of negacyclic matrices, order 3 for the Goethals-Seidel array

$1,1,1,1$	$a00,$	$000,$	$c00,$	$d00$	(as for cyclic)
$1,1,1,4$	$a00,$	$b00,$	$cdd,$	$0d\bar{d}$	
$1,1,1,9$	$add,$	$bdd,$	$cdd,$	$d\bar{d}d$	
$1,1,2,2$	$a00,$	$b00,$	$cd0,$	$c\bar{d}0$	(as for cyclic)
$1,1,2,8$	$add,$	$bdd,$	$cd\bar{d},$	$c\bar{d}d$	
$1,1,4,4$	$acc,$	$ac\bar{c},$	bdd	$0b\bar{b}$	
$1,1,5,5$	$acc,$	$bdd,$	$cd\bar{d},$	$c\bar{d}c$	
$1,2,2,4$	$ad\bar{d},$	$0d\bar{d},$	$bc0,$	$b\bar{c}0$	
$1,2,3,6$	$add,$	$cd\bar{d},$	$c\bar{d}b,$	$c\bar{d}\bar{b}$	
$2,2,2,2$	$ab0,$	$a\bar{b}0,$	$cd0,$	$c\bar{d}0$	(as for cyclic)
$2,2,4,4$	$a\bar{c}d,$	$ac\bar{d},$	$bc\bar{d},$	$b\bar{c}d$	
$3,3,3,3$	abc	$\bar{b}ad$	$\bar{c}\bar{d}a$	$\bar{d}c\bar{b}$	(as for cyclic)

3 Applications

In [1], 4-suitable negacyclic matrices are used to construct new orthogonal bipolar spreading sequences for any length $4 \pmod 8$ where the resultant sets of sequences possess very good autocorrelation properties that make them amenable to synchronization requirements. In particular, their aperiodic autocorrelation characteristics are very good.

It is well known, e.g. [23, 25], that if the sequences have good aperiodic cross-correlation properties, the transmission performance can be improved for those CDMA systems where different propagation delays exist. Orthogonal bipolar sequences are of a great practical interest for the current and future direct sequence (DS) code-division multiple-access (CDMA) systems where the orthogonality principle can be used for channels separation, e.g. [1]. The most commonly used sets of bipolar sequences are Walsh-Hadamard sequences [23], as they are easy to generate and simple to implement. However, they exist only for sequence lengths which are an integer power of 2, which can be a limiting factor in some applications. The overall autocorrelation properties of the modified sequence sets are still significantly better than those of Walsh-Hadamard sequences of comparable lengths.

3.1 Combinatorial Applications

For combinatorial applications see [10, 3, 2, 19, 17].

We also see from papers [11, 12, 13, 14] that OD's in orders 24, 40, 48, 56, 80, that had proved difficult to constructed using circulant matrices were found using negacyclic matrices.

References

[1] R. Ang, J. Seberry, B.J. Wysocki, and T.A. Wysocki. Application of nega-cyclic matrices to generate spreading sequences. In *Proceedings of the Seventh International Symposium on Communications Theory and Applications (ISCTA2003)*, Ambleside, UK, July 2003. ISCTA2003, HW Communications Limited.

[2] Gerald Berman. Families of skew-circulant weighing matrices. *Ars Combinatoria*, 4:293–307, 1977.

[3] Gerald Berman. Weighing matrices and group divisible designs determined by $EG(t, p^r)$, $p > 2$. *Utilitas Math.*, 12:183–191, 1977.

[4] P. Delsarte, J. M. Goethals, and J. J. Seidel. Orthogonal matrices with zero diagonal II. *Canad. J. Math.*, 23:816–832, 1971.

[5] Peter Eades. *On the Existence of Orthogonal Designs*. PhD Thesis, Australian National University, Canberra, Australia, 1977.

[6] S. Georgiou, W.H. Holzmann, H. Kharaghani, and B. Tayfeh-Rezaie. Some tables for: Three variable full orthogonal designs of order 56. University of Lethbridge: http://www.cs.uleth.ca/ holzmann/cgi-bin/ODall.pl/table.pdf, 2007.

[7] S. Georgiou, W.H. Holzmann, H. Kharaghani, and B. Tayfeh-Rezaie. Three variable full orthogonal designs of order 56. *Journal of Statistical Planning and Inference*, 137(2):611–618, 2007.

[8] A. V. Geramita and J. Seberry. *Orthogonal Designs: Quadratic forms and Hadamard matrices*, volume 45 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker, 1st edition, 1979.

[9] J. M. Goethals and J. J. Seidel. Orthogonal matrices with zero diagonal. *Canad. J. Math.*, 19:1001–1010, 1967.

[10] J. M. Goethals and J. J. Seidel. A skew-Hadamard matrix of order 36. *J. Austral. Math. Soc.*, 11:343–344, 1970.

[11] Wolf H Holzmann and Hadi Kharaghani. On the orthogonal designs of order 24. *Discrete Applied Mathematics*, 102(1-2):103–114, 2000.

[12] Wolf H Holzmann and Hadi Kharaghani. On the orthogonal designs of order 40. *Journal of Statistical Planning and Inference*, 96(2):415 – 429, 2001.

[13] Wolf H Holzmann, Hadi Kharaghani, Jennifer Seberry, and Behruz Tayfeh-Rezaie. On orthogonal designs in order 48. *Journal of Statistical Planning and Inference*, 128(1):311 – 325, 2005.

[14] Wolf H Holzmann, Hadi Kharaghani, and Behruz Tayfeh-Rezaie. All triples for orthogonal designs of order 40. *Discrete Mathematics*, 308(13):2796–2801, 2008.

[15] N. Ito. On Hadamard groups III. *Kyushu J. Math.J*, 51:369–379, 1997.

[16] N. Ito. On Hadamard groups IV. *Journal of Algebra*, 234:651663, 2000.

[17] Zvonimir Janko and Hadi Kharaghani. A block negacyclic Bush-type Hadamard matrix and two strongly regular graphs. *Journal of Combinatorial Theory, Series A*, 98(1):118–126, 2002.

[18] Hadi Kharaghani. Arrays for orthogonal designs. *J. Combin. Designs*, 8(3):166–173, 2000.

[19] Hadi Kharaghani. On a class of symmetric balanced generalized weighing matrices. *Designs, Codes and Cryptography*, 30(2):139–149, 2003.

[20] Hadi Kharaghani and Behruz Tayfeh-Rezaie. Some new orthogonal designs in orders 32 and 40. *Discrete Mathematics*, 279(1-3):317 – 324, 2004.

[21] Jennifer Seberry and N. A. Balonin. The propus construction for symmetric Hadamard matrices. to appear.

[22] Edward Spence. Hadamard matrices from relative difference sets. *J. Combinatorial Theory*, Ser. A,(19):287–300, 1975.

[23] P. Vial, B.J. Wysocki, I. Raad, and T. Wysocki. Space time spreading with modified Walsh-Hadamard sequences. In *Spread Spectrum Techniques and Applications, 2004 IEEE Eighth International Symposium on*, pages 943–946, Aug 2004.

[24] Jennifer Seberry Wallis and Albert Leon Whiteman. Some classes of Hadamard matrices with constant diagonal. *Bull. Austral. Math. Soc.*, 7:233–249, 1972.

[25] B. J. Wysocki and T. A. Wysocki. Modified Walsh-Hadamard sequences for DS CDMA wireless systems. *International Journal of Adaptive Control and Signal Processing*, 16(8):589–602, 2002.

[26] Tianbing Xia, Ming-Yuan Xia, and Jennifer Seberry. Hadamard matrices constructed from circulant and nega-cyclic matrices. *Australasian J. Comb.*, 34:105–116, 2006.