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A NOTE ON ORDERED SHEAVES

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Abstract—*Ordered vector spaces in which the cone plays an important role is significant in the study of spectral theory of positive operators. The Lattice Structures as linearly compact simplexes arise from algebraic properties of cones. Ordered topological vector spaces combine the concept of order convergence and topological convergence. The study of Germs and Sheaves of sets and in particular that of analytic functions on $\mathbb{C}_n$ has given rise to numerous classical situations as well as applications in different areas of mathematics. Interrelating the concepts of partial order relation and sheaves of sets, the idea of ordered sheaves of sets has been introduced in this paper.*

Keywords: *Order Topology, Sheaf of Sets*

PRELIMINARIES [1]

Definition1.1: An ordered vector space is a real vector space E equipped with a transitive, reflexive, antisymmetric relation ≤ satisfying the following conditions:

1. If x, y, z are elements of E and $x \leq y$, then $x + z \leq y + z$.
2. If x, y, z are elements of E and α is a positive real number, then $x \leq y$ implies $\alpha x \leq \alpha y$:

The notation $y \geq x$ often will be used in place of $x \leq y$. If $x \leq y$ and $x \neq y$,

we shall write $x < y$.

The positive cone (or simply the cone) C in an ordered vector space E is defined by

$C = \{x \in E : x \geq 0\}$, where 0 denotes the zero element in E. The cone K has the following properties:

1. $C + C \subset C$
2. $\alpha C \subset C$ for each positive real number α.
3. $C \cap (-C) = \{0\}$.

In particular, If C is a convex set in E satisfying (1), (2) and (3), then:

$x \leq y$ if $y - x \in C$ defines an order relation $\leq$ on E with respect to which E is an ordered vector space with positive cone C. A subset W of E containing 0 and satisfying (1) and (2) is called a wedge.

Let E be an ordered vector space. If x, y are elements of E and $x \leq y$, then the set $[x,y] = \{z \in E : x \leq z \leq y\}$ is the order interval between x and y. A subset B of E is order bounded if there exist x, y in E such that $B \subset [x,y]$. A subset D of E is majorized (resp. minorized) if there is an element z in E such that $z \geq d$ (resp. $z \leq d$) for all $d \in D$. If every pair x,y of elements of a subset D is majorized (minorized) in D, then D is directed(≤) [resp.directed(≥)].

Definition 1.2: If A is a subset of a vector space E ordered by a cone C, the **full hull** [A] of A is defined by

$[A] = \{z \in E : x \leq y \leq z \text{ for } x \in A, y \in A\}$

That is, [A] = (A + K) ∩ (A – K). If A = [A], then A is full.

Definition 1.3: Suppose that E($\mathcal{F}$) is an ordered topological vector space and that C is the positive cone in E($\mathcal{F}$). C is **normal** for the topology $\mathcal{F}$ if there is a neighbourhood basis of 0 for $\mathcal{F}$ consisting of full sets.

Example: Examples of normal cones in ordered topological vector spaces are plenty.

1. The cone C = {x = (x_k) ∈ $\mathbb{R}^n$: x_k ≥ 0} for k = {1, 2, ……} is normal for the Euclidean topology on $\mathbb{R}^n$.
2. The cone of non-negative functions in the space C(X) of continuous, real valued functions on a compact Hausdorff space X, or in the space B(X) of bounded real valued functions on the set X, is normal for the topology generated by the norm

||f|| = sup{|f(t)| : t ∈ X}

ORDERING IN THE COMPLEX NUMBER SYSTEM

The complex plane X is a vector space. Define a relation '≤' on X in the following way: z_1 = a+ib and z_2 = c+id be two complex numbers where a, b, c, d are real numbers then $z_1 \le z_2$ if a ≤ c and b ≤ d. Then '≤' is a transitive, reflexive, antisymmetric relation satisfying the following properties:

1. If x, y, z are elements of X and x≤y, then x + z ≤ y + z.
2. If x, y are elements of X and α is a positive real number, then x≤ y implies αx≤αy.

Thus C is an ordered vector space.

ORDERED SHEAVES

Definition 3.1: An ordered sheaf S =(S,,X) of abelian groups is an order preserving onto map π:S→X, where S and X are ordered topological spaces, such that

1. π is a local homeomorphism and order preserving.
2. for each x∈X, π^{-1}(x) an ordered abelian group.
3. Addition is continuous w.r.t order topology on S.

That π is a local homeomorphism and order preserving means that for each point p∈S, there is an open set G (in the order topology of S) such that π/G maps G homeomorphically onto some open set π(G) (in the order topology of S)

S is called ordered sheaf space, π is the order preserving projection map and X is the ordered base space.

Result 3.2: The open sets of S which project homeomorphically onto open sets of X form a base for open sets in order topology of S.

Proof: If p is in a open set H,there exist an order interval G,p∈G such that H∩G maps G homeomorphically onto π(G). Then H∩G is an order interval, p∈ H∩G ⊂H and π/ H∩G maps H∩G homeomorphically onto (H∩G),an order interval and hence open in X.

***Result 3.3*:** π is an order preserving continuous mapping.

***Proof*:** Continuity of π follows from the fact that π is a local homeomorphism and from result 1, π is an open mapping.

***Definition 3.4*:** The set $S_x= \pi^{-1}(x)$ is called the ordered stalk of S at x. S_X is an abelian group. If (p)≠ π(q),then p+q is not defined.

***Result 3.5*:** S_X has the order topology, the relative topology w.r.t the order in S.

***Proof*:** This is a consequence of the fact that π is a local homeomorphism and order preserving.

Let S x S be the Cartesian product of the ordered space S with itself and let S+S be the subspace consisting of those pairs (p,q) for which $\pi(p)= \pi(q)$.

Addition is continuous means that f: S+S $\rightarrow$ S defined by f(p,q)=p+q is continuous i.e. if p,q$\in$ S and $\pi(p)= \pi(q)$ then given an order interval G containing p+q, there exist order interval H,K with p$\in$H, q$\in$K such that if r$\in$H, s$\in$K and $\pi(r)= \pi(s)$,then r+s$\in$G. i.e H+K$\subset$G.

Result 3.6

1. Zero and inverse are continuous.

Writing O_x for the zero element of the group $S_{X,}$zero is continuous means that f: X$\rightarrow$S, where f(x)= O_x is continuous.

Writing –p for the inverse of p in the ordered group $S_{(p)}$, Inverse is continuous. That is g:S$\rightarrow$S where g(p)=-p is continuous.

Example 3.7

1. Ordered sheaf of topological vector spaces
2. Ordered sheaf of analytic function elements

Let X be the sphere of ordered complex numbers. Let S_X be the ordered additive group of function elements at x, each function element being a power series converging w.r.t the order topology in some order interval containing x. Let S=$\cup_x S_X$ and define π:S$\rightarrow$X by $\pi(S_X)$=x.An order relation is defined on the set G of all germs f_x with x$\in$D, an open set in X, by $f_x \leq f_y$ iff x ≤ y in D. Then $\mathcal{G}$ is an ordered topological vector space. If x is a function element, a neighbourhood of x in S is defined by analytic continuation f_x. Then $\mathcal{S}$ =(S,,X) is the ordered sheaf of analytic function elements. Each component of S is a Riemann surface without branch points. The ordered sheaf $\mathcal{S}$ is Hausdorff.

DIRECTED SECTIONS

Definition 4.1: A directed section of an ordered sheaf $\mathcal{S}$ =(S, X) over an open set U$\subset$X is an order preserving continuous function f:U$\rightarrow$S such that $\pi\cdot$f=I/U where I/U denotes the identity function on U.

The image f(U) is also called a directed section.

***Result 4.2*:** For each open set U$\subset$X the order preserving map f: U$\rightarrow$S where, f(x)= O_x is a directed section.

***Proof*:** The map f(x) = O_x is continuous and (O_x)=x and both f and π are order preserving. This section is called directed 0-section.

***Example 4.3*:** If $\mathbb{S}$ =(S,π,X) is the ordered sheaf of function elements over the complex sphere X, the set of all directed sections of $\mathbb{S}$ over a non-empty open set U forms an ordered abelian group Γ(U, $\mathbb{S}$). W.r.t the usual operations of addition and multiplication of functions Γ(X, $\mathbb{S}$) can be identified with the ordered ring of functions analytic in U. Then Γ(X, $\mathbb{S}$) can be identified with the ordered ring of functions, analytic everywhere, hence is isomorphic to the ordered ring of complex numbers

***Result 4.4*:** A directed section f:U$\rightarrow$S is an open mapping.

***Proof*:** f is order preserving, continuous and π is order preserving and a local homeomorphism.

***Result 4.5*:** The necessary and sufficient condition that a set G $\subset$ S is a directed section f(U) over some open set U $\subset$X (in the order topology of X) is that G is open and π/G is a homeomorphism.

Proof: Let f be a directed section over U. Since f is open, f(U) is open. Also f from U to f(U) is one to one, open, continuous π/f(U) : f(U) $\rightarrow$ U is a homeomorphism. Converse follows from definition of directed section.

***Result 4.6*:** The directed sections f(U) form a base for the open sets of S(in the order topology of S).

***Proof*:** Result 8 shows that if f is a directed section over U then f is a homeomorphism of U onto f(U).Result 1 shows that the open sets of S which project homeomorphically onto open sets of X form a base for the open sets of S in the order topology of S. Thus the directed sections f(U) form a base for the open sets of S in the order topology.

***Result 4.7*:** The intersection f(U)$\cap$g(V) of two directed sections is a directed section.

***Proof*:** f(U)$\cap$g(V) is open in the order topology of S and projects homeomorphically onto an open set of X(in the order topology of X),since each of f(U) and g(V) project homeomorphically onto open set of X(in the order topology of X).

Remark 4.8

The order topology of S=$\cup_x S_X$ may be described by specifying the directed sections since they form a base for the open sets of S in the order topology.

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